Laying the Groundwork for Tensor Network Approaches to Lattice Yang-Mills Theory

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Perturbation theory renders a plethora of processes computable in quantum field theory. Nevertheless, certain phenomena are inherently non-perturbative, particularly in the case of QCD and confinement. Lattice gauge theory is one avenue which allows one to make due without perturbative expansions. In this framework, path integrals reduce to statistical mechanical computations equivalent to those of the canonical ensemble, and the integration measure is well-defined. One can then use Markov Chain Monte Carlo (MCMC) approaches to calculate expectations and correlation functions. In general, the presence of fermionic Dirac spinors on the lattice can sometimes lead to a sign problem and prevent these algorithms from converging. Instead of using the law of large numbers to compute numerical values, the present work focuses on simplifying the expression of the partition function of pure gauge lattice Yang-Mills theory (YM). To start with, we introduce the techniques used here by applying them to the well-known Ising model. We then state pure gauge YM on the lattice, and address the continuum limit and renormalization of such a theory. In order to study confinement without resorting to numerics, we also look at an approximation involving decoupled plaquettes, which provides a toy model of confinement – or the lack thereof – in arbitrary dimensions. We then focus on expressing partition functions as tensor networks, in abelian and non-abelian theories, using representation theory, and incidentally rederive some results concerning 1+1-dimensional YM. We subsequently show that the local L^2 error for a given tensor is subpolynomial in a power of the ratio between inverse temperature and bond dimension. Understanding the tensor network formulation of lattice YM will enable the use of algorithms which do not resort to sign-problem-riddled MCMC, and might provide a step forward in understanding confinement and the mass gap problem.

Preface

The work presented here is the result of a three-and-a-half-month internship in the QUANTIC group at Inria Paris, under the supervision of Prof. Antoine Tilloy. One of the long-term goal of the group is to simulate phenomena involving the strong interaction using tensor networks. My short-term goal was to go as far as I could in this direction, starting with learning basic QFT and lattice gauge theory, and deriving (probably rederiving in most cases) elementary results which yield qualitative insights into the behavior of abelian and non-abelian theories, as well as phase transitions and confinement.

One of my main contributions is probably concisely building lattice gauge theories from the ground up, as seen through the eyes of a master's student, which I hope yields a pedagogical treatment of the subject matter. Before resorting to numerics, I also tried to push analytical results as far as I could, hence the focus on approximations and 1+1-dimensional theories in a significant part of my work. More closely related to the task at hand, one contribution is the expression of the relevant tensor networks in the special case of YM, whether abelian or non-abelian, which resembles formulae given in previous work on lattice theories, except perhaps for the use of conventional lattice QCD notation. Moreover, I've studied the local error induced by the truncation of the possible bond values in the tensor networks, which is a key element in the numerical implementation of tensor network algorithms.

I'd like to thank Prof. Antoine Tilloy for his supervision and for granting me much of his time and attention. His support and guidance have also been vital to my continued studies as an aspiring theoretical physicist. Thanks are also in order to Karan Tiwana and Dr. Edoardo Lauria, for invaluable discussions surrounding fascinating physics, and for making QFT more accessible to someone such as myself who has yet to take all of the relevant classes.

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1 Introduction

Yang and Mills introduced the concept of non abelian gauge theory to generalize the abelian theory describing electromagnetism [1]. The framework they developed, Yang-Mills theory (YM), describes the behavior of gauge fields, which are fundamental to the understanding of fundamental forces in particle physics except for gravity. QCD was shown to fit the YM landscape by Gell-Mann and Fritzsch, who proposed the idea of quark color charges and the use of the $SU(3)$ local symmetry group [2]. This led to the current picture we have of QCD: fermions known as quarks endowed with a color, interacting via exchange of massless gauge bosons known as gluons, which also self-interact. In particular, these self-interactions are an immediate consequence of the non-abelian nature of the gauge group of QCD.

Further contributions led to YM being used for computations in QCD. 't Hooft and Veltman showed how to renormalize non-Abelian gauge theories [3], which was necessary to making meaningful calculations and predictions. Another significant advancement came from the discovery of asymptotic freedom in non-abelian gauge theories by Gross, Wilczek and Politzer. They showed that the strong force becomes weaker at high energies, allowing for a perturbative approach to calculations in QCD [4, 5].

Nevertheless, at lower energy scales, perturbation theory breaks down, and one needs non-perturbative methods. A popular avenue is lattice gauge theory, the foundations of which were laid down through the works of Wilson, Kogut and Susskind, who introduced the concept of lattice gauge theory and its application to QCD [6, 7], the latter only discretizing space and keeping time continuous. Lattice gauge theory involves discretizing spacetime into a lattice, with the fields and particles living on the lattice points. This allows for a rigorous treatment of non-perturbative aspects of gauge theories, such as confinement and the generation of particle masses, as well as predictions of asymptotic freedom. Moreover, lattice QCD has been instrumental in investigating the properties of hadrons, QCD phase transitions, and the behavior of matter under extreme conditions.

Lattice gauge theory also has a role to play in mathematical physics. Indeed, it might provide a rigorous approach to constructing quantum field theories. There have been efforts to axiomatize QFT and verify that typical field theories – such as YM – satisfy certain axioms. Two famous examples are the Wightman axioms [8], expressed using Lorentzian time, and the Osterwalder-Shrader axioms for Euclidean time [9]. Moreover, the path integral formulation – the more explicitly Lorentz-invariant method for quantizing field theories – uses integration measures over paths in the continuum which aren't rigorously defined in the general case. Resorting to the lattice and the associated Haar measure, and subsequently taking the continuum limit, is one approach to constructing a rigorous path integral.

While lattice gauge theory has benefitted from the advancement of high-performance computing, computing observables in lattice gauge theory can still be extremely costly, both in terms of time and spatial complexity. As such, it is useful to borrow techniques from tensor networks to compress certain tensors appearing in various computations. For all intents and purposes, tensors are defined as multidimensional arrays of numbers here. As for tensor networks, they are a mathematical framework used to describe and manipulate contractions of potentially high-dimensional tensors [10]. They provide a way to efficiently represent and compute with large sets of data or complex quantum states. One commonly used tensor network state, the simplest in practice, is the Matrix Product State (MPS). MPS provides a representation of quantum states in one dimension, which is particularly useful for simulating systems with one-dimensional structures, such as spin chains. More generally, one can also work with Projected Entangled Pair States (PEPS). PEPS extend the MPS framework to higher dimensions, making it suitable for simulating two-dimensional entanglement.

Tensor networks are particularly useful to the study of phase transitions in various field theories, particularly when coupled to coarse-graining methods associated with the renormalization group. These have been successfully used to tackle the first models of quantum field theory, such as ϕ^4 theory [11].

Research has also been conducted in quantum gravity which uses lattice approaches and tensor networks by Dittrich et al. The cases of finite or abelian gauge groups is extensively studied [12, 13]. Nevertheless, non-abelian gauge theory remains elusive, and as such YM still isn't fully understood on the lattice.

2 A First Example

Let's start with a simpler lattice theory whose study involves all of the elements we will use for YM in the present work: the Ising model. While it is not a gauge theory (there is global, as opposed to local $\mathbb{Z}_2$ symmetry), representation theory is still a powerful tool in extracting results from its partition function. In the context of the canonical ensemble, let's define the Ising (nearest neigbor) Hamiltonian to be

$$H = -J \sum_{<i,j>} x_i x_j \tag{2.1}$$

where the spin variables are taken to be in $\mathbb{U}_2 \cong \mathbb{Z}_2$ and we'll take $J = 1$ to only consider inverse temperature as a parameter of the theory. We'll be assuming periodic boundary conditions here, as well as in all of the subsequent lattice models. The associated partition function is given by

$$Z = \sum_{\{x\}} \prod_{<i,j>} e^{\beta x_i x_j} \tag{2.2}$$

The mapping $f : x \in \mathbb{U}_2 \mapsto e^{\beta x} \in \mathbb{R}_+^*$ is trivially (for an abelian group) a central function i.e. for a group element g, $f(gxg^{-1}) = f(x)$ for all x. The Peter-Weyl theorem [14] allows us to write the following expansion of f (this is also trivial here, but will be worth noting in non-abelian gauge theories):

$$f = \sum_r c_r \chi_r \tag{2.3}$$

where the χ_r are the irreducible group characters of $\mathbb{U}_2$. These are simply defined by $\chi_r((-1)^k) = (-1)^{kr}$ for $r \in \{0, 1\}$. The c_r are the associated Fourier coefficients given by

$$\begin{aligned} c_r &= \frac{1}{2} \sum_{x \in \{-1,1\}} f(x) \chi_r(x) \\ &= \frac{1}{2} \left(e^{\beta} + (-1)^r e^{-\beta} \right) \\ &= \cosh(\beta) \exp(r \ln \tanh \beta) \end{aligned} \tag{2.4}$$

Since the gauge group here is abelian, the characters are multiplicative. Thus

$$\begin{aligned} Z &= \sum_{\{x\}} \prod_{<i,j>} \sum_r c_r \chi_r(x_i) \chi_r(x_j) \\ &= \sum_{\{r\}} \left(\prod_{<i,j>} c_{r_{<i,j>}} \right) \sum_{\{x\}} \prod_{<i,j>} \chi_{r_{<i,j>}}(x_i) \chi_{r_{<i,j>}}(x_j) \\ &= \sum_{\{r\}} \left(\prod_{<i,j>} c_{r_{<i,j>}} \right) \prod_i \sum_{x_i} \prod_{j \in <i,.>} \chi_{r_{<i,j>}}(x_i) \\ &= 2^N \sum_{\{r\}} \left(\prod_{<i,j>} c_{r_{<i,j>}} \right) \prod_i \delta_2 \left(\sum_{j \in <i,.>} r_{<i,j>} \right) \end{aligned} \tag{2.5}$$

Where the sum inside the δ_2 is to be taken modulo 2.

2.1 One-dimensional Case

For $d = 1$, the δ_2 constraints amount to summing over configurations where the character indices are identical at every interaction. Therefore

$$\begin{aligned} Z &= 2^N \left(\cosh^N(\beta) + \sinh^N(\beta) \right) \\ &= 2^N \cosh^N(\beta) \left(1 + \tanh^N(\beta) \right) \\ &\underset{N \to \infty}{\approx} 2^N \cosh^N(\beta) \end{aligned} \tag{2.6}$$

This clearly yields an analytical free energy, which implies the lack of any phase transition, in accordance with Landau theory.

2.2 Two-dimensional Case

The two-dimensional case is significantly more complicated, but nonetheless tractable. Each link of the spin lattice carries an interaction, and is characterized by a lattice site n and a direction μ. We can thus rewrite the partition function (2.5) the following way

$$Z = 2^N \sum_{\{r\}} \left(\prod_{n,\mu} c_{r_{n\mu}} \right) \prod_n \delta_2 \left(\sum_\mu (r_{n\mu} + r_{(n-\mu)\mu}) \right) \tag{2.7}$$

We construct a dual lattice which satisfies the δ constraints by definition [15]. The dual lattice has vertices at the center of each unit cell. We attribute a value $\sigma_m \in \{-1, 1\}$ to each vertex, where m is the dual lattice site. To each link $r_{n\mu}$of the original lattice, we can uniquely associate a pair of dual variables $\sigma_m, \sigma_{m-\nu}$, with $\mu \neq \nu$. We can choose to write

$$r_{n\mu} = \frac{1}{2} (1 - \sigma_m \sigma_{m-\nu}) \tag{2.8}$$

One can check that this satisfies the δ constraints by definition. Expressing the partition function using the dual variables, we arrive at

$$Z = 2^N \sum_{\{\sigma\}} \prod_{m,\nu} c_{\frac{1}{2}(1-\sigma_m\sigma_{m-\nu})} \tag{2.9}$$

Plugging in (2.4), we obtain

$$\begin{aligned} Z &= 2^N \cosh^N (\beta) \sinh^N \beta \\ &\times \sum_{\{\sigma\}} \exp \left(-\frac{1}{2} \ln \tanh (\beta) \sigma_m \sigma_{m-\nu} \right) \end{aligned} \tag{2.10}$$

Which can also be rewritten

$$Z_\beta = \frac{1}{\sinh^N 2\tilde{\beta}} Z_{\tilde{\beta}} \tag{2.11}$$

Notice that this proportional to a partition function for an two-dimensional Ising model on the dual lattice if we define a new inverse temperature $\tilde{\beta} = -\frac{1}{2} \ln \tanh (\beta)$. The model is said to be self-dual (in the sense of Kramers-Wannier duality). We can show that this model presents a critical point at $\beta = \tilde{\beta}(\beta)$ [15], which yields the following critical parameter:

$$\beta_c = \frac{\ln (1 + \sqrt{2})}{2} \tag{2.12}$$

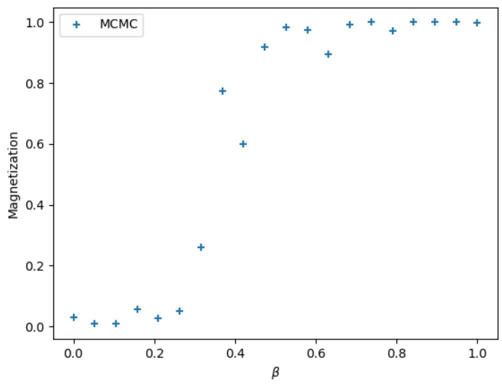

Figure 1: Magnetization of a two-dimensional spin lattice as a function of inverse temperature. The plot is generated using the Metropolis-Hastings algorithm.

This is the first example of a phase transition, which is an important concept in lattice gauge theory.

2.3 Mean Field Approximation

If the number of dimensions is high enough, $d \geq 4$ for the Ising model, we can model nearest neighbor interactions using an effective interaction. Indeed, the coupling between neighboring spins is weaker if the cardinality of its neighborhood, $2d$, is big. We can thus approximate the real Hamiltonian by the following (taking a coupling $J = 1$):

$$H = - \sum_{<i,j>} x_i \langle x \rangle \tag{2.13}$$

This leads to a simple partition function, $Z = 2^{n^d} \cosh^{n^d} (2\beta d \langle x \rangle)$. Calculating the average spin magnetization leads to the following identity

$$\langle x \rangle = \tanh (2\beta d \langle x \rangle) \tag{2.14}$$

Let's consider this as the equality between two functions of $\langle x \rangle$, the identity and a function involving tanh. For small β, $\langle x \rangle = 0$ is the only intersection between the two functions' graphs, whereas for large

β, other intersections are possible. The critical inverse temperature is obtained when the slope of the two functions are equal:

$$\beta_c = \frac{1}{2d} \tag{2.15}$$

We will use these concepts for lattice YM, after having introduced and summarized the construction of the aforementioned theory in the following section.

3 Stating Pure Yang-Mills Theory on the Lattice

In this section, we derive the equivalent of the following Euclidean YM action on the lattice, largely following the steps outlined in [16]:

$$S = \frac{1}{4g^2} \int \mathcal{D}\left[A_\mu\right] \operatorname{Tr} F_{\mu\nu} F^{\mu\nu} \tag{3.1}$$

3.1 Fermion Action

As in the continuum, deriving gauge fields relies on defining a gauge covariant derivative in order to construct a gauge invariant Dirac fermion action. Consider a four-dimensional hypercubical lattice Λ of lattice constant a. As in the continuum case, we'd like the action to be locally G-invariant for a gauge group G. For all $n \in \Lambda$, let $\Omega(n) \in G$. Consider that under the action of G, we have the following transformation $\psi(n) \to \psi'(n) = \Omega(n)\psi(n)$. We need to make the discrete derivative term covariant. One way to do this is to place directional link variables $U_\mu(n)$ on each link in the lattice, and consider the following identity, where the primed variables are transformed from the unprimed ones following a gauge transformation:

$$\begin{aligned} &\overline{\psi}'(n)U'_\mu(n)\psi'(n+\hat{\mu}) \\ &= \overline{\psi}(n)\Omega(n)^\dagger U'_\mu(n)\Omega(n+\hat{\mu})\psi(n+\hat{\mu}) \end{aligned} \tag{3.2}$$

This expression can be made covariant provided that

$$U_\mu(n) \to \Omega(n)U_\mu(n)\Omega(n+\hat{\mu})^\dagger \tag{3.3}$$

We also define the link variable in the opposite direction using $U_{-\mu}(n) := U_\mu(n-\mu)^\dagger$. Let's finally define the discrete covariant derivative in the μ direction:

$$D_\mu\psi(n) = \frac{U_\mu(n)\psi(n+\hat{\mu}) - U_{-\mu}(n)\psi(n-\hat{\mu})}{2a} \tag{3.4}$$

By construction, the following Euclidean action is G-invariant:

$$S_F = a^d \sum_{n\in\Lambda} \overline{\psi}(n)\left(\gamma^\mu D_\mu\psi(n) + m\psi(n)\right)) \tag{3.5}$$

where the $a^d\Sigma$ term stands for a discrete integral.

3.2 The Wilson Gauge Action

By construction, the following path-ordered product is clearly gauge-invariant [16]:

$$W_{\mathcal{L}} = \operatorname{Tr}\left[\prod_{(n,\mu)\in\mathcal{L}} U_\mu(n)\right] \tag{3.6}$$

where $\mathcal{L}$ is a closed loop, referred to as a Wilson loop. By analogy with the continuum case, we use the shortest nontrivial closed loop on the lattice, called a plaquette. We define this path by:

$$\begin{aligned} U_{\mu\nu}(n) &= U_\mu(n)U_\nu(n+\mu)U_{-\mu}(n+\mu+\nu)U_{-\nu}(n+\nu) \\ &= U_\mu(n)U_\nu(n+\mu)U_\mu(n+\nu)^\dagger U_\nu(n)^\dagger \end{aligned} \tag{3.7}$$

We can then use the following gauge boson action:

$$S_G = \frac{2}{g^2}\sum_{n\in\Lambda}\sum_{\mu<\nu} \Re(\operatorname{Tr}\left[I - U_{\mu\nu}(n)\right]) \tag{3.8}$$

3.3 Continuum Limit and Renormalization

In the continuum limit, for $a \to 0$, the Wilson gauge action ought to go the continuum Yang-Mills action. Indeed, we can introduce the gauge fields $A_\mu(n)$ such that

$$U_\mu(n) = e^{iaA_\mu(n)} \tag{3.9}$$

Using the BCH identity and expanding the displaced gauge fields in a, we get [17]:

$$\begin{aligned} S_G &\approx \frac{a^4}{2g^2} \sum_{n\in\Lambda} \sum_{\mu<\nu} \operatorname{Tr} F_{\mu\nu}^2 \\ &\approx \frac{a^4}{4g^2} \sum_{n\in\Lambda} \operatorname{Tr} F_{\mu\nu} F^{\mu\nu} \end{aligned} \tag{3.10}$$

In four dimensions, we retrieve the continuum YM action:

$$S_G \approx \frac{1}{4g^2} \int \operatorname{Tr} F_{\mu\nu} F^{\mu\nu} \tag{3.11}$$

Otherwise, we get a constant in front of the YM continuum action depending on the lattice constant. For d dimensions, this constant is

$$\gamma = \frac{a^{4-d}}{g^2} \tag{3.12}$$

In order to make sure the lattice theory goes to the continuum as $a \to 0$ when $d \neq 4$, we have to force γ to be equal to a constant by taking a coupling $g(a)$ defined by:

$$g(a) = g_0 a^{2-\frac{d}{2}} \tag{3.13}$$

The discrete theory converges to the continuum theory as a goes to zero provided we take $g(a)$ as the coupling constant. Thus defined, the coupling has dimensions of energy to the $(d/2-2)$-th power. It is therefore marginal for $d = 4$, relevant for $d < 4$ and irrelevant for $d > 4$ [18]. The continuum limit will be reached for $\beta \to +\infty$ when $d < 4$ and $\beta \to 0$ when $d > 4$ a priori. Nevertheless, this statement for $d > 4$ is dubious, since it doesn't account for the non-renormalizability of the associated continuum theory. We do not discuss this further here.

We can now define the continuum limit in the context of the present work. We'll be considering a hypercubic box of size L^d, and taking a number of lattice points such that $na(\beta) = L$. This roughly amounts to taking

$$n = L\left(\beta g_0^2\right)^{\frac{1}{4-d}} \tag{3.14}$$

In what follows, we will only work with the Euclidean gauge action S_G, and discard the fermion action S_F. We'll also be using $g_0 = 1$.

3.4 The Yang-Mills Lattice Path Integral

Using Euclidean time, one can reformulate the discrete field-theoretic path integral as a statistical mechanical theory of random matrices. Indeed, the discrete path integral yields expectation values for operators O of the form

$$\langle O\rangle = \frac{1}{Z} \int \mathcal{D}[U] O[U] e^{-S_G[U]} \tag{3.15}$$

where we define a partition function

$$Z = \int \mathcal{D}[U] e^{-S_G[U]} \tag{3.16}$$

and the following measure over the configuration of link variables:

$$\mathcal{D}[U] = \prod_{n\in\Lambda} \prod_{\mu=1}^{4} dU_\mu(n) \tag{3.17}$$

where $dU_\mu(n)$ is the Haar measure defined on G, which must be a compact group [14] for it to be defined. This statistical mechanical theory is stated in the canonical ensemble, with an inverse temperature $\beta = 1/g^2$. The action acts as an energy times an inverse temperature, thus we'll be calculating, analytically and numerically, the following quantity:

$$\langle S\rangle = -\beta \frac{\partial \ln Z}{\partial \beta} \tag{3.18}$$

3.5 The Wilson Loop Order Parameter

It can be shown that the expectation values of individual link variables, and more generally local parameters, cannot be used as order parameters, contrary to the case of the Ising model [19]. Nevertheless, non-local parameters, such as Wilson loops, are good indicators of potential phase transitions. Therefore, on top of looking at partition functions and expectation values of the action, we shall be studying expectations of Wilson loops. It turns out that Wilson loops are correlators of Wilson lines. Consider a loop in the $t-x$ plane. We can use the so-called "temporal gauge" and set the link variables in the temporal direction equal to the identity matrix (or 1 in abelian gauge theory). Thus, a loop is the trace of the product of two Wilson lines, which we suppose to be taken at times $t=0$ and $t=an_t$. Let's call these lines $S(m,n,0)$ and $S(m,n,n_t)$ respectively, where m,n are the endpoints of a given line, the loop in question being a square for the sake of this argument (this notation does not carry over to the following sections, where S exclusively refers to an action). Their Euclidean correlator yields the expectation of the Wilson loop [16]:

$$\begin{aligned}\langle W_{\mathcal{L}}\rangle &= \langle \operatorname{Tr}\left[S(m,n,n_t)S(m,n,0)^\dagger\right]\rangle \\ &= \sum_k \langle 0|S(m,n)_{ab}|k\rangle \langle k|S(m,n)_{ba}|0\rangle\, e^{-tE_k}\end{aligned} \tag{3.19}$$

Where the operators on the right are taken in the Schrödinger picture. It can be shown that $S(m,n)^\dagger_{ba}|0\rangle$ corresponds to a quark-antiquark pair in the heavy quark limit. The lowest energy term in the sum corresponds to a static quark-antiquark pair, and the subsequent terms can be interpreted as additional particle-antiquark pairs. If we ignore higher order terms, we're left with

$$\langle W_{\mathcal{L}}\rangle \propto e^{-tV(r)} \tag{3.20}$$

where V is the potential of the strong interaction in the quark-antiquark pair. In the next sections, we'll be calculating $\langle W_{\mathcal{L}}\rangle$ in order to extract $V(r)$, and study the form of the potential. In particular, a linearly increasing potential is a sign of confinement.

4 Approximations

4.1 Decoupled Plaquette Approximation

Now that we've introduced YM theory on the lattice, we can start digging for some analytical results – which mainly involve a low-dimensional or approximate configuration. We're looking at the latter in this section. Analogously to the Ising model, one can study lattice YM in what naively resembles mean field approximation. Indeed, one way to introduce the mean field in the Ising model is to consider uncorrelated spins. We can do the same for YM, by decoupling plaquettes. Quite surprisingly, it turns out that 1+1-dimensional YM is virtually indistinguishable from the decoupled plaquette approximation we define here (we shall give reasons for this when exploring 1+1-dimensional gauge theory), and we can extract some of the qualitative behavior of QCD from the model. Here, we simplify Haar integrals by integrating over plaquettes instead of link variables, thus ignoring the "overlap" between different plaquettes. Invariance by action on the right in the Haar measure yields $dU_\mu(n) = dU_{\mu\nu}(n)$. Thus, distributing differential group elements in the product defining the partition function becomes simple:

$$\begin{aligned}Z &= \left(\int dU e^{2\beta(\Re \operatorname{Tr} U - 2N)}\right)^{\frac{d(d-1)n^d}{2}} \\ &= f(\beta)^{\frac{d(d-1)n^d}{2}}\end{aligned} \tag{4.1}$$

where the integral defining f is well known, since it is closely proportional to modified Bessel functions of the first kind. Indeed, it is given by [20, 21]:

$$\begin{cases} f_{SU(N)}(\beta) = \sum_{l\in\mathbb{Z}} \det\left[I_{l+j-i}(2\beta)\right]_{1\le i,j\le N} \\ f_{U(N)}(\beta) = \det\left[I_{j-i}(2\beta)\right]_{1\le i,j\le N}\end{cases} \tag{4.2}$$

where for $n\in\mathbb{Z}$, $x\in\mathbb{R}$

$$I_n(2x) = e^{-2Nx}\sum_{k=0}^{+\infty}\frac{x^{2k+n}}{k!\Gamma(k+n+1)} \tag{4.3}$$

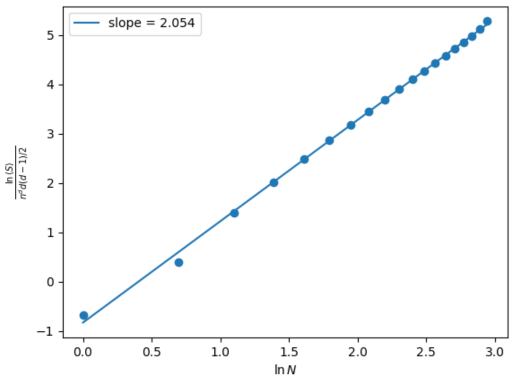

Figure 2: Regression for large β behavior yielding N^2 scaling of $\langle S\rangle$. The points are calculated using (4.1).

It is worth noticing that I_n is identical to the Fourier coefficients determined for $U(1)$. Moreover, numerical calculations show (see figure 2) that for large β

$$\langle S\rangle \propto N^2 n^d d(d-1) \tag{4.4}$$

In particular, the N^2 scaling is comparable to a result derived for 1+1 dimensions in continuum $U(N)$ theory by Chatterjee [22].

As for Wilson loops, for a given loop $\mathcal{L}$ defined by

$$\langle W_{\mathcal{L}}\rangle = \Re\langle \mathrm{Tr} \prod_{(n,\mu)\in\mathcal{L}} U_\mu(n)\rangle \tag{4.5}$$

we can consider the set $\mathcal{A}$ of plaquettes contained in $\mathcal{L}$ and write

$$\langle W_{\mathcal{L}}\rangle = \Re\langle \mathrm{Tr} \prod_{(n,\mu,\nu)\in\mathcal{A}} U_{\mu\nu}(n)\rangle \tag{4.6}$$

This is clear graphically in figure 4.

The calculation of the expectation value yields a factor of Z (due to the plaquettes which are outside of the loop). Therefore:

$$\langle W_{\mathcal{L}}\rangle = f(\beta)^{-n_P}\, \Re\,\mathrm{Tr}\left[J(\beta)^{n_P}\right] \tag{4.7}$$

where J has the following integral expression:

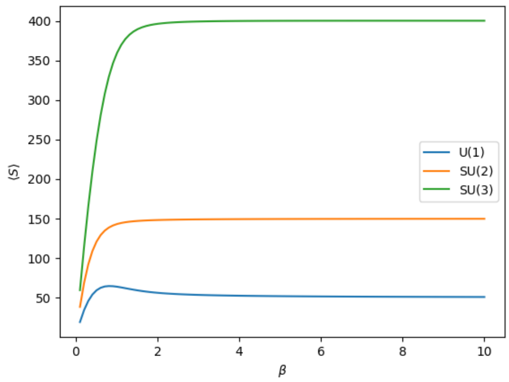

Figure 3: Average action as function of β for $U(1)$, $SU(2)$ and $SU(3)$.

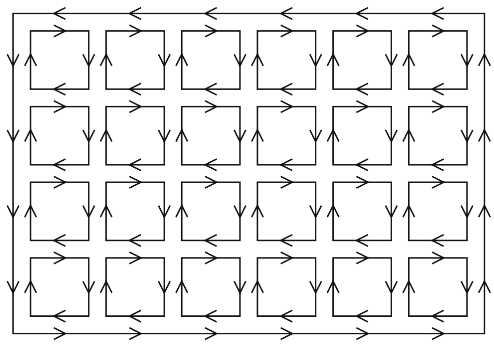

Figure 4: The product of outer link variables is equal to the product of plaquettes inside the loop. Figure taken from [16], where it is labeled figure 3.4.

$$[J(\beta)]_{ij} = \int dU U_{ij} e^{2\beta(\Re \operatorname{Tr} U - N)} \tag{4.8}$$

This integral can be computed using Creutz's method [23], which is also concisely summarized by Carlsson in [21]. Roughly speaking, this method draws on the fact that Haar integrals of polynomials in U and $U^\dagger$ for $U \in SU(N)$ are polynomials in Levi-Civita symbols and Kronecker deltas, whose coefficients are derived using known integrals. Using this method, it can be shown that the integral is proportional to a pair of Levi-Civita symbols (Carlsson calculates a very similar integral):

$$[J(\beta)]_{ij} \propto \epsilon_{ia_1 \dots a_{N-1}} \epsilon_{ja_1 \dots a_{N-1}} \propto \delta_{ij} \tag{4.9}$$

We therefore use the ansatz $[J(\beta)]_{ij} = h(\beta)\delta_{ij}$. We can then determine h using a known integral. Indeed, tracing out the Kronecker δ yields

$$h(\beta) = \frac{1}{N} \int dU \operatorname{Tr} U e^{2\beta(\Re \operatorname{Tr} U - N)} \tag{4.10}$$

To evaluate this integral, we borrow notation from Carlsson [21]:

$$h(\beta) = \frac{1}{N} \frac{\partial G}{\partial c}\bigg|_{c=d=\beta} \tag{4.11}$$

where G is a generating function defined in Carlsson's paper. This leads to the following result:

$$h(\beta) = \frac{1}{N} \sum_{l \in \mathbb{Z}} \operatorname{Tr}\left[\operatorname{adj}\left(\{I_{l+i-j}(2\beta)\}\right) \cdot \{I'_{l+i-j}(2\beta)\}\right] \tag{4.12}$$

Therefore, Wilson loops are given by

$$\langle W_{\mathcal{L}} \rangle = N \left(\frac{f(\beta)}{h(\beta)}\right)^{-n_P} \tag{4.13}$$

If $f(\beta) > h(\beta)$, this gives rise to confinement. Indeed, using the definition of the potential V from Gattringer and Lang [16], we get a linear potential:

$$V(r) = \frac{1}{a^2} \ln\left(\frac{f(\beta)}{h(\beta)}\right) r \tag{4.14}$$

For $d \neq 4$, renormalization actually yields

$$V(r) = \left(g_0^2 \beta\right)^{\frac{2}{4-d}} \ln\left(\frac{f(\beta)}{h(\beta)}\right) r = \sigma(\beta) r \tag{4.15}$$

where we introduce the string tension σ. Therefore, whether or not confinement is present in the continuum limit depends on d a priori. Numerical calculations for $U(1)$ and $SU(N)$ for $N = 2$ and 3 show that for $d = 2$ or 3, confinement is present in the continuum, which corresponds to weak coupling.

For $d > 4$, the continuum limit is also confined, since the continuum corresponds to strong coupling. However, this continuum limit is different from the true continuum YM theory in $d > 4$ dimensions, since the latter is non-renormalizable. As a final remark on $d > 4$, it is worth noting that the string constant goes to zero at large distances (large β according to (3.13)), suggesting deconfinement

For $d = 4$, a is decoupled from β. For finite β, taking $a \to 0$ leads to confinement. In short, it seems the theory is always confining for the continuous gauge groups studied here.

4.2 Strong coupling

The limit $\beta \to 0$ is worth studying, since it can be obtained through simple computations, and also corresponds to a continuum limit for $d > 4$. Clearly, for $\beta \to 0$

$$\begin{aligned} Z &= \int \mathcal{D}[U] e^{-S} \\ &= 1 - 2\beta \Re \sum_{n,\mu<\nu} \int \left(N - \operatorname{Tr} U_{\mu\nu}(n)\right) \mathcal{D}[U] + o(\beta) \end{aligned} \tag{4.16}$$

Expanding the traces of the plaquette terms, using the Schur orthogonality relations relative to the trivial representation, we finally arrive at

$$Z = 1 - d(d-1) N n^d \beta + o(\beta) \tag{4.17}$$

which reduces to the expression derived previously for $d = 2$, and yields a linear average gauge action at strong coupling.

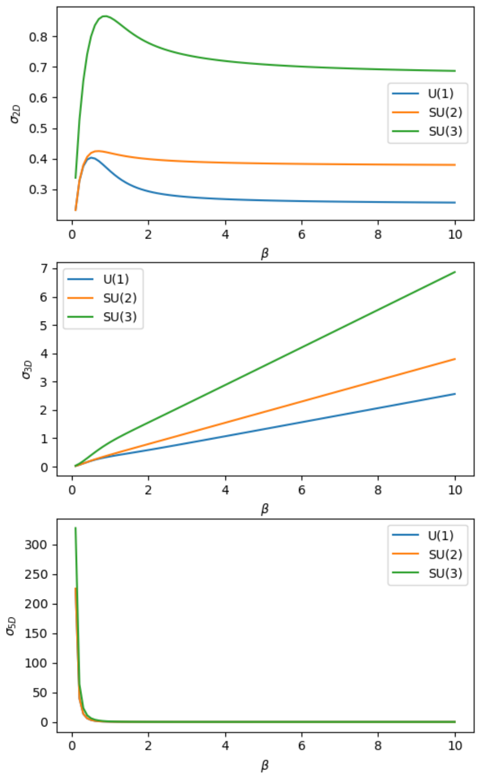

Figure 5: String constant (in units of energy squared) for different gauge groups in 2, 3 and 5 dimensions from top to bottom.

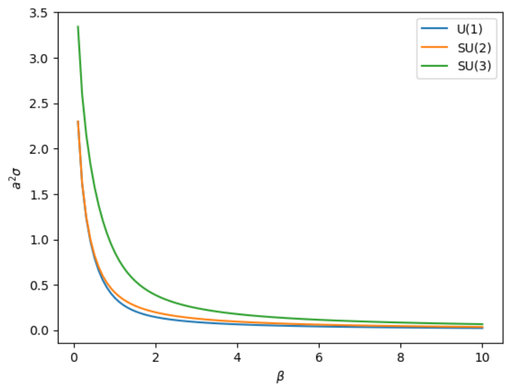

Figure 6: String constant times lattice spacing squared as a function of β.

In the next sections, we will be dropping the aforementioned approximations and try to approach the YM action as is.

5 Abelian Gauge Theory

5.1 Partition Function

Due to its importance in QFT and particle physics, we start by studying $U(1)$ theory in the pure gauge regime, which amounts to working on lattice QED. The trace in the action given previously in (3.8) can be removed, yielding the following simplified action

$$S_G = 2\beta \sum_{n\in\Lambda} \sum_{\mu<\nu} \Re(\mathrm{Tr}\left[I - U_{\mu\nu}(n)\right]) \tag{5.1}$$

The partition function can then be written

$$Z = \int \mathcal{D}[U] \prod_{\substack{n\in\Lambda \\ \mu<\nu}} \exp\left(-2\beta\left(1 - \Re\left(U_{\mu\nu}(n)\right)\right)\right) \tag{5.2}$$

Next, we apply the procedure used to study the Ising model's partition function. Consider the function $f : U \in U(1) \mapsto \exp\left(-2\beta\left(1 - \Re\left(U\right)\right)\right) \in \mathbb{R}_+^*$.

This is a central function on $U(1)$, and as such it is subject to the theory of harmonic analysis on compact groups, generalizing the theory on finite groups [14]. The irreducible representations of $U(1)$ are characterized by individual integers, hence the following expansion

$$f = \sum_{r \in \mathbb{Z}} c_r \chi_r \tag{5.3}$$

where χ_r are the characters of irreducible representations of $U(1)$ i.e. $\chi_r(e^{i\theta}) = e^{-ir\theta}$. The Fourier coefficients c_r are defined by

$$\begin{aligned} c_r &= \frac{1}{2\pi} \int_0^{2\pi} f\left(e^{i\theta}\right) \overline{\chi}_r\left(e^{i\theta}\right) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{2\beta(\cos\theta - 1)} e^{ir\theta} d\theta \end{aligned} \tag{5.4}$$

Incidentally, these coefficients can be evaluated using quadrature, or using the following expansion, derived using the Taylor expansion of the exponential and the linearization of monomials in $\cos\theta$:

$$c_r = \sum_{j=0}^{+\infty} \frac{\beta^{2j+r} e^{-2\beta}}{j!(j+r)!} \tag{5.5}$$

These coefficients are proportional to modified Bessel functions of the first kind. Next, we plug this expansion into the integrand $I[U]$ in the definition of the partition function and use the multiplicativity of the group characters:

$$\begin{aligned} I[U] = & \prod_{n,\mu<\nu} \sum_r c_r \chi_r(U_\mu(n)) \chi_r(U_\nu(n+\mu)) \\ & \times \overline{\chi}_r(U_\mu(n+\nu)) \overline{\chi}_r(U_\nu(n)) \\ = & \sum_{\{r\}} \prod_{n,\mu<\nu} c_{r_{n\mu\nu}} \chi_{r_{n\mu\nu}}(U_\mu(n)) \chi_{r_{n\mu\nu}}(U_\nu(n+\mu)) \\ & \times \overline{\chi}_{r_{n\mu\nu}}(U_\mu(n+\nu)) \overline{\chi}_{r_{n\mu\nu}}(U_\nu(n)) \end{aligned} \tag{5.6}$$

We get the final result by using periodic boundary conditions and rearranging the product:

$$\begin{aligned} Z &= \int \mathcal{D}[U] I[U] \\ &= \sum_{(r)} \left(\prod_{n,\mu<\nu} c_{r_{n\mu\nu}} \right) \\ &\times \left(\prod_{n,\mu} \int \prod_{\nu=\mu+1}^{d-1} \chi_{r_{n\mu\nu}} \overline{\chi}_{r_{(n-\nu)\mu\nu}} \prod_{\nu=0}^{\mu-1} \chi_{r_{(n-\nu)\nu\mu}} \overline{\chi}_{r_{n\nu\mu}} \right) \end{aligned} \tag{5.7}$$

In fact, these integrals can be simplified using the expressions for $U(1)$ characters, yielding the following result:

$$Z = \sum_{(r)} \left(\prod_{n,\mu<\nu} c_{r_{n\mu\nu}} \right) \prod_{n,\mu} \delta\left(\Delta^\nu r_{n\mu\nu}\right) \tag{5.8}$$

where we use the Einstein summation convention (without distinctions between covariant and contravariant), take $r_{n\mu\nu} = -r_{n\nu\mu}$ and a discrete derivative defined by

$$\Delta_\nu r_{n\alpha\beta} = r_{n\alpha\beta} - r_{(n-\nu)\alpha\beta} \tag{5.9}$$

Following the previous reasoning on the Ising model, we can define a dual lattice, wherein the new variables $\phi_{n\mu} \in \mathbb{Z}$ are on the previous lattice links, rather than the plaquettes:

$$r_{n\mu\nu} = \begin{cases} \varepsilon_{\nu\lambda} \Delta^\lambda \phi_{n\mu} \text{ if } \mu < \nu \\ -\varepsilon_{\nu\lambda} \Delta^\lambda \phi_{n\mu} \text{ if } \mu > \nu \end{cases} \tag{5.10}$$

where we've used Levi-Civita symbols $\varepsilon_{\mu\nu\lambda}$. Thus, the dual of $U(1)$ is $\mathbb{Z}$, and the theory is not self dual [24]. One can easily check that this new lattice satisfies the δ constraints, and we obtain the following partition function:

$$Z = \sum_{\{\phi\}} \prod_{n,\mu<\nu} c_{\varepsilon_{\mu\nu} \varepsilon_{\mu\nu\lambda} \Delta^\lambda \phi_n} \tag{5.11}$$

Notice that this doesn't converge a priori, but this doesn't matter in practice since the partition function can be defined up to an infinite uniform factor [15], and we'll also be using a cutoff on the variables.

5.2 Tensor Network

The partition function can be written as a tensor network, in at least two different ways depending on the previous section. Indeed, we can use the dual lattice or not. Let's ignore the duality transformation for the moment. One way to get a tensor network is to define the following tensors:

$$T_{\{r_{n\mu\nu}\}} = \left(\prod_{\kappa<\lambda} c^{\frac{1}{d}}_{r_{n\kappa\lambda}}\right) \delta\left(\Delta^{\nu} r_{n\mu\nu}\right) \tag{5.12}$$

In d dimensions, this tensor is a $2(d-1)$-legged tensor living on a lattice link, contracted with all tensors sharing a plaquette with it.

Next, we use the L^2 norm to define the error induced by ignoring all irreps of hyperparameter beyond a certain bound. Let's define the set S of allowed irreducible representations $S = \{r|\forall(n,\mu,\nu), r_{n\mu\nu} \leq D\}$. Then the local error is given by

$$\varepsilon = \sum_{\{r\}\notin S} T^2_{n\mu} \tag{5.13}$$

Clearly

$$\varepsilon \leq \sum_{\{r\}} \prod_{\mu<\nu} c^{\frac{2}{d}}_{r_{n\mu\nu}} \tag{5.14}$$

An asymptotic estimate of this error can be derived using the integral expression for the Fourier coefficients and using integration by parts, integrating the real exponential term j times. Indeed, integration by parts yields

$$\begin{aligned} c_r &= \frac{e^{-2\beta}}{2\pi}\int_0^{2\pi} e^{2\beta\cos\theta} e^{ir\theta} d\theta \\ &= \frac{(-1)^j e^{-2\beta}}{2\pi i r^j}\int_0^{2\pi} \left[e^{2\beta\cos\theta}\right]^{(j)} e^{ir\theta} d\theta \end{aligned} \tag{5.15}$$

Expanding the j-th derivative in the integrand in powers of β/r leads to

$$c_r = \mathcal{O}\left(\left(\frac{\beta}{r}\right)^j\right) \tag{5.16}$$

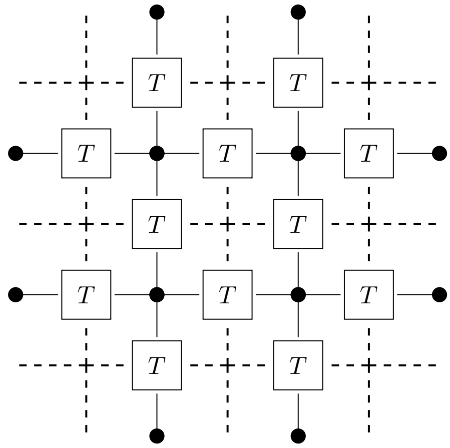

Figure 7: Tensor network representing the partition function of 1+1-dimensional lattice (abelian) gauge theory. The tensors are placed on the links of the spacetime lattice (dashed lines), and are contracted with tensors on the corresponding plaquettes. The black dots stand for Kronecker δ's.

Therefore

$$\prod_{\mu<\nu} c^{\frac{2}{d}}_{r_{n\mu\nu}} = \mathcal{O}\left(\beta^{j(d-1)} \prod_{\mu<\nu} \frac{1}{r^j_{n\mu\nu}}\right) \tag{5.17}$$

Next, we sum over all configurations of r with at least one $r > D$. The dominant term is the one for which all $r = D$. Thus, we arrive at

$$\varepsilon = \mathcal{O}\left(\left(\frac{\beta}{D}\right)^{j(d-1)}\right) \tag{5.18}$$

which is valid for all j: the error is subpolynomial in $(\beta/D)^{d-1}$.

Introducing the duality transformation leads to tensors living on the plaquettes themselves. Those tensors are instead the $c_{\varepsilon_{\nu\lambda}\Delta^{\lambda}\phi_{n\mu}}$ previously mentioned, which are also $2(d-1)$-legged tensors. In d dimensions, the network is a result of the product of independent $d-1$-dimensional networks.

5.3 Expectation of a Wilson Loop

The expectation value of a Wilson loop can also be expressed using a tensor network, once we have the partition function:

$$\langle W_{\mathcal{L}} \rangle = \frac{1}{Z} \int \mathcal{D}[U] W_{\mathcal{L}}[U] e^{-S_G[U]} \tag{5.19}$$

In the abelian case, the Wilson loop reduces to

$$W_{\mathcal{L}} = \prod_{(n,\mu)\in\mathcal{L}} U_\mu(n) = \prod_{(n,\mu)\in\mathcal{L}} \chi_{-1}\left(U_\mu(n)\right) \tag{5.20}$$

This means that the numerator in (5.19) closely resembles the partition function Z, except for a few tensors. Let's call the tensors appearing in the numerator $V_{\{r_{n\mu\nu}\}}$. These tensors are defined by

$$V_{\{r_{n\mu\nu}\}} = \begin{cases} T_{\{r_{n\mu\nu}\}} \text{ if } (n,\mu) \notin \mathcal{L} \text{ and } (n+\mu,-\mu) \notin \mathcal{L} \\ \left(\prod_{\kappa<\lambda} c_{r_{n\kappa\lambda}}^{\frac{1}{d}}\right) \delta\left(\varepsilon_{\mu\nu}\Delta^\nu r_{n\mu\nu} \pm 1\right) \text{else} \end{cases} \tag{5.21}$$

where the sign is positive for $(n,\mu) \in \mathcal{L}$ and negative for $(n+\mu,-\mu) \in \mathcal{L}$ in the second case. Then the final result is given by

$$\langle W_{\mathcal{L}} \rangle = \frac{1}{Z} \sum_{\{r\}} \prod_{n,\mu} V_{\{r_{n\mu\nu}\}} \tag{5.22}$$

5.4 1+1-Dimensional Case

In a 2D lattice, the partition function is given by

$$Z = \sum_{r\in\mathbb{Z}} c_r^{n^2} \tag{5.23}$$

In fact, $c_r = c_{-r}$, so that we actually get

$$Z = c_0^{n^2} + 2\sum_{r=1}^{+\infty} c_r^{n^2} \tag{5.24}$$

This expression allows one to derive the following expression in the strong coupling limit:

$$Z = 1 - 2n^2\beta + o(\beta) \tag{5.25}$$

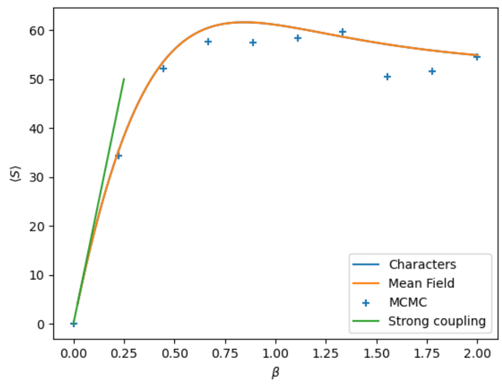

Figure 8: Average Wilson $U(1)$ action in the two-dimensional case, plotted using character expansion and Markov chain Monte Carlo method.

This yields a linear average action, which is confirmed in the simulations for $U(1)$. We'll see that this is also the case for $SU(N)$ for $N = 2, 3$.

Let's consider a Wilson loop enclosing n_P plaquettes. The δ constraints can be solved for easily in 1+1 dimensions. Indeed, we can take a representation parameter r inside the loop, and $r-1$ outside. This yields:

$$\langle W_{\mathcal{L}} \rangle = \frac{1}{Z} \sum_{r\in\mathbb{Z}} c_r^{n_P} c_{r-1}^{n^2-n_P} \tag{5.26}$$

This gives rise to confinement. It is worth comparing this expression with the one derived in the mean field/decoupled plaquette approximation defined previously. Indeed, the first term is identical to the mean field approximation (if we take $Z \approx c_0^{n^2}$, which is reasonable per numerical simulations), while additional terms represent a deviation arising from the "overlap" between plaquettes. Numerical results suggest that these additional terms are negligible (see figure (8)). Moreover, they can be attributed physical meaning, using equation 3.19. Indeed, the additional terms account for energy contributions from extra particle/antiparticle pairs. The deviation in higher dimensions from the decoupled plaquette model sug-

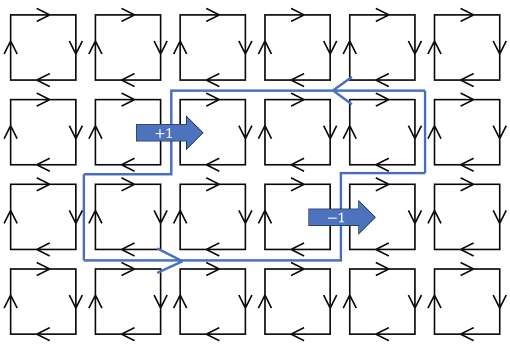

Figure 9: Wilson loop in 1+1-dimensions. The representation parameter inside the loop is equal to the parameter outside of it, plus one.

gests that we might be ignoring too many plaquette overlaps (a plaquette shares link variables with $8(d-1)-4 = 4(2d-3)$ other plaquettes).

5.5 Finite Gauge Theory

To finish the section on abelian theories, we can consider approximating continuous gauge groups using finite groups. As it turns out, $\mathbb{Z}_N$ can approximate $U(1)$ theory when $N \to +\infty$. This is an idea that can be generalized to the study of quantum groups replacing the usual gauge groups. These quantum groups can be seen as "deformed", or sometimes discretized versions of the original group. The partition function for $\mathbb{Z}_N$ is the following:

$$Z = \sum_{\{x\}} \prod_{n,\mu<\nu} e^{2\beta(x_{\mu\nu}(n)-1)} \tag{5.27}$$

A character expansion leads to [13]:

$$Z = N^{dn^d} \sum_{\{r\}} \left(\prod_{n,\mu<\nu} c_{r_{n\mu\nu}} \right) \prod_{n,\mu} \delta_N \left(\Delta^\nu r_{n\mu\nu} \right) \tag{5.28}$$

where δ_N is equal to one if its argument is equal to zero modulo N. The integrals defining Fourier coefficients for infinite groups are replaced by sums with a factor of $1/N$. This yields the following coefficients:

$$c_r = e^{-2\beta} \sum_{m=0}^{+\infty} \sum_{l=0}^{+\infty} \frac{\beta^{2m+r+lN}}{m!\,(m+r+lN)!} \tag{5.29}$$

For large N, these coefficients go to those of $U(1)$ theory. In fact, this can be seen in the string constants plotted in figure 10.

Moreover, mean $\mathbb{Z}_N$ theory is free in the continuum limit for $d < 4$, contrary to mean $U(1)$ theory. Nevertheless, the continuum limit (small β) is still confined for $d \geq 4$.

The treatment of non-abelian YM, introduced in the next section, follows the same steps, despite additional difficulties arising from a more complicated group structure.

6 Non-Abelian Gauge Theory

6.1 Partition Function

In the non-abelian case, the group character are no longer multiplicative, and the group integrals are nontrivial. For a given irreducible representation π, the associated character is defined using $\chi_\pi(x) = \operatorname{Tr} \pi(x)$ for all x. Let's define c_π to be the Fourier component relative to $\pi \in \hat{G}$ of the function $f : U \in G \mapsto \exp\left(2\beta(\Re \operatorname{Tr}(U - I))\right) \in \mathbb{R}_+^*$. Then the partition function takes the following form:

$$Z = \sum_{\substack{(\pi_{\mu\nu n}) \\ (i),(j),(k),(l)}} \left(\prod_{n,\mu<\nu} c_{\pi_{n\mu\nu}} \right) \times \left(\int \prod_{\nu=\mu+1}^{d-1} \pi^{ij}_{n\mu\nu} \overline{\pi}^{kl}_{(n-\nu)\mu\nu} \prod_{\nu=0}^{\mu-1} \overline{\pi}^{li}_{n\nu\mu} \pi^{jk}_{(n-\nu)\nu\mu} \right) \tag{6.1}$$

where we've expanded the traces defining representation characters, and introduced matrix index variables i, j, k, l which are integer-valued fields defined over plaquettes, and represent additional variables to contract over (these yield finite bond dimensions however, contrary to representations). In the formula above, i, j, k, l are evaluated at the same plaquette as

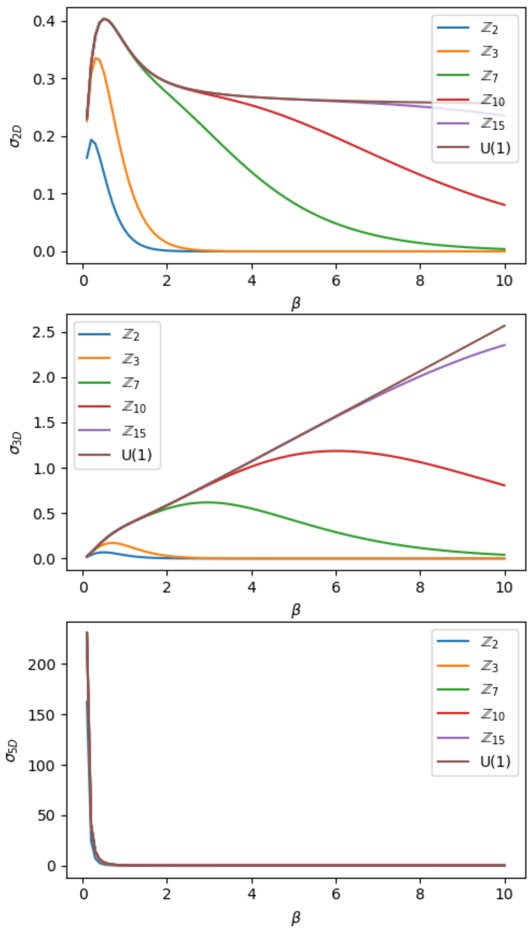

Figure 10: Mean $\mathbb{Z}_N$ gauge theoretic string constant σ for $d = 2, 3, 5$ compared to $U(1)$ theory.

the π representation. One of the difficulties of non-abelian theories on the lattice is that the Haar integrals don't have simpler expressions, using a Dirac δ or Levi-Civita symbols, which could lead to the use of duality methods [12]. Nonetheless, we can still express the partition function as a tensor network, and study the induced local error.

6.2 Tensor Network

The tensors are very similar to the ones derived in the abelian case, except for the nontrivial group integrals involving representation matrix elements:

$$T_{\{\pi_{n\mu\nu}\}} = \left(\prod_{\kappa<\lambda} c_{\pi_{n\kappa\lambda}}^{\frac{1}{d}}\right) \int \prod_{\nu=\mu+1}^{d-1} \pi_{n\mu\nu}^{ij} \overline{\pi}_{(n-\nu)\mu\nu}^{kl} \times \prod_{\nu=0}^{\mu-1} \overline{\pi}_{n\nu\mu}^{li} \pi_{(n-\nu)\nu\mu}^{jk} \tag{6.2}$$

Furthermore, we have introduced additional tensor indices, since we are working with matrix elements instead of exclusively working with characters (this was necessary because of the non-multiplicativity of non-abelian group characters). As such, these tensors are $6(d-1)$-legged, and non-abelian theories induce more intricate tensor networks. That being said, one can recover a topology identical to the abelian case, by concatenating "representation" and "matrix index" variables into a single multi-index, and replacing the Kronecker δ's on plaquettes from the abelian theory with a more intricate tensor (this can easily be achieved by taking products of Kronecker δ's).

Estimating the local error in the tensor network is slightly more involved here. We will neglect truncation over the i, j, k, l variables since they induce a finite bond dimension N (nevertheless this might be worth considering carefully for large N). We can use the following formula stated in [21]:

$$c_{r_1 \ldots r_{N-1}} = e^{-2\beta N} \sum_{l\in\mathbb{Z}} \det I_{j-i+l+n_i}(2\beta) \tag{6.3}$$

Hadamard's inequality (applied to the transpose) yields an upper bound on the determinants in the summand:

$$\det I_{j-i+l+n_i}(2\beta) \leq \prod_{j=1}^{N} \sqrt{\sum_{i=1}^{N} I_{i-j+l+n_j}(2\beta)^2} \quad (6.4)$$

Since the bond dimension D is taken to be reasonably large, and $n_i > D$ in the error calculation, we can assume that the $i-j$ indices don't affect the upper bound that much. Thus

$$\det I_{j-i+l+n_i}(2\beta) \leq N^{\frac{N}{2}} \prod_{j=1}^{N} I_{l+n_j}(2\beta)^2 \quad (6.5)$$

We therefore have

$$c_{r_1 \ldots r_{N-1}} \leq N^{\frac{N}{2}} e^{-2\beta N} \sum_{l\in\mathbb{Z}} \prod_{j=1}^{N} I_{l+n_j}(2\beta)^2 \quad (6.6)$$

Moreover, since modified Bessel functions of the first kind increase as their parameter goes to zero (for positive argument), and $n_N := 0$, the term that dominates will be that for which the quantity

$$l^2 + \sum_{j=1}^{N-1} (l+n_j)^2 \quad (6.7)$$

is minimized i.e. $l = -\frac{1}{N}\sum_{j=1}^{N-1} n_j$. We now have an asymptotic bound

$$c_{r_1 \ldots r_{N-1}} = \mathcal{O}\left(e^{-2\beta N} \prod_{j=1}^{N} I_{n_j - \frac{1}{N}\sum_{k=1}^{N-1} n_k}(2\beta)^2\right) \quad (6.8)$$

The dominating term in the sum defining the local error is that for which all parameters are equal to the bond dimension D. This yields, using the asymptotic bound for $I_n(2x)$ obtained using integration by parts:

$$\varepsilon = \mathcal{O}\left(\left(\frac{\beta}{D}\right)^{j(N-1)(d-1)}\right) \quad (6.9)$$

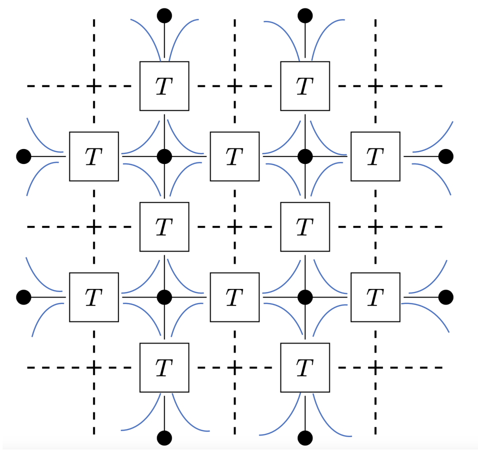

Figure 11: Tensor network representing the partition function of 1+1-dimensional lattice (non-abelian) gauge theory. The tensors are placed on the links of the spacetime lattice (dashed lines), and are contracted with tensors on the corresponding plaquettes. The solid lines represent contractions over representations, the black dots stand for Kronecker δ's, while the blue arcs are contractions over matrix element indices.

Again, the error is subpolynomial, this time in $(\beta/D)^{(N-1)(d-1)}$, and as observed numerically, higher β requires more computational power, since more terms need to be included.

6.3 Expectation of a Wilson Loop

The procedure for calculating expectation values of Wilson loops is identical to that described for abelian YM. As for the abelian case, only a few tensors are modified in the network. For this section, let's write Wilson loops in the following way:

$$W_{\mathcal{L}} = \sum_{i_1,\ldots,i_{|\mathcal{L}|-1}} \prod_{k=1}^{|\mathcal{L}|-1} [U_{\mu_k}(n_k)]_{i_k,i_{k+1}} \quad (6.10)$$

where we define $\mu_{|\mathcal{L}|} = \mu_1$ and $n_{|\mathcal{L}|} = n_1$. Elementary tensors are now given by:

$$V_{\{\pi_{n\mu\nu}\}} = \begin{cases} T_{n\mu} \text{ if } (n,\mu) \notin \mathcal{L} \text{ and } (n+\mu,-\mu) \notin \mathcal{L} \\ \left(\prod_{\kappa<\lambda} c_{\pi_{n\kappa\lambda}}^{\frac{1}{d}}\right) I_{n\mu} \text{ otherwise} \end{cases} \tag{6.11}$$

where

$$I_{n\mu} = \int \mathrm{id}_{i_k,i_{k+1}} \prod_{\nu=\mu+1}^{d} \pi_{n\mu\nu}^{ij} \overline{\pi}_{(n-\nu)\mu\nu}^{kl} \prod_{\nu=1}^{\mu-1} \overline{\pi}_{n\nu\mu}^{li} \pi_{(n-\nu)\nu\mu}^{jk} \tag{6.12}$$

In the above integral, $\mathrm{id}_{i_k,i_{k+1}}$ should be understood as the function which maps a link variable to its (i_k, i_{k+1}) component.

6.4 1+1-Dimensional Case

For any Yang-Mills pure gauge theory, the partition function takes the following form in 1+1 dimensions:

$$Z = \sum_{\pi\in\hat{G}} \left(\frac{c_\pi}{\dim\pi}\right)^{n^2} \tag{6.13}$$

where $\hat{G}$ is the set of irreducible representations of G. It's worth noting that this expression is consistent with $U(1)$ two-dimensional theory, since its abelian nature leads to $\dim\pi = 1$ for all π, according to Schur's lemma.

The formula for Wilson loop expectation values is also more involved than for $U(1)$:

$$\langle W_{\mathcal{L}}\rangle = \frac{1}{Z} \sum_{\pi,\rho,i,j} \left(\frac{c_\pi}{\dim\pi}\right)^{n_P} \left(\frac{c_\rho}{\dim\rho}\right)^{n^2-n_P} \prod_{(n,\mu)\in\mathcal{L}} I_{n\mu} \tag{6.14}$$

This displays an area law, a strong hint of confinement, which is confirmed numerically. Since $SU(N)$ for $N = 2$ or 3 are groups of particular note, we illustrate these results in the next subsections for these groups.

6.4.1 $SU(2)$ Case

For $G = SU(2)$, we arrive at

$$Z = \sum_{r=0}^{+\infty} \left(\frac{c_r}{r+1}\right)^{n^2} \tag{6.15}$$

Indeed, irreducible representations are characterized by a single largest weight r. They are $(r+1)$-dimensional, and have the following characters [25]:

$$\chi_r(\theta) = \frac{\sin((r+1)\theta)}{\sin\theta} \tag{6.16}$$

where θ is the equivalent of the spherical polar angle for the 3-sphere. Indeed, $SU(2) \cong S^3$, and as such group elements are characterized by 3-sphere angles in spherical coordinates [26, 27]. We can also get an analytical expression for the Fourier coefficients in the same way as in the abelian case:

$$c_r = e^{-4\beta} \sum_{k=\lceil -\frac{r}{2}\rceil} \frac{2^{2k-1+r} a_{rk}}{(2k+r)!} \beta^{2k+r} \tag{6.17}$$

where we define

$$a_{rk} = \binom{r+2k}{k} + \binom{r+2k}{-k} - \binom{r+2k}{k-1} - \binom{r+2k}{-(k-1)} \tag{6.18}$$

As before, in the strong coupling limit, we can approximate the partition function by an affine function. For $\beta \to 0$

$$Z = 1 - 4n^2\beta + o(\beta) \tag{6.19}$$

6.4.2 $SU(3)$ Case

For $SU(3)$, we instead get

$$Z = \sum_{p,q=1}^{+\infty} \left(\frac{2c_{pq}}{pq(p+q)}\right)^{n^2} \tag{6.20}$$

Here, we've chosen expressions for the characters defined using two of the eigenvalues of a group element A, B [28]:

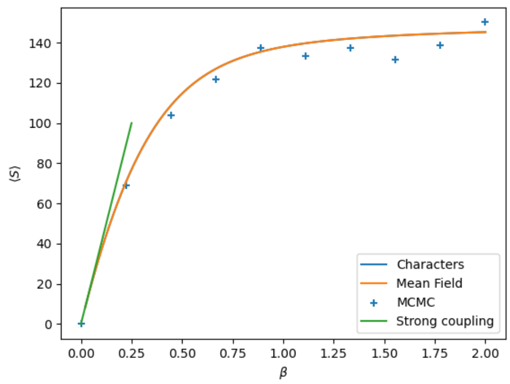

Figure 12: Average Wilson $SU(2)$ action in the two-dimensional case, plotted using character expansion and Markov chain Monte Carlo method.

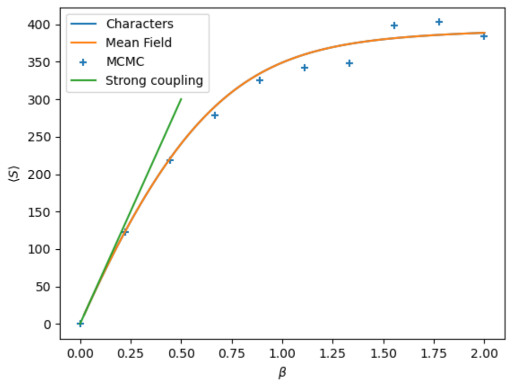

Figure 13: Average Wilson $SU(3)$ action in the two-dimensional case, plotted using character expansion and Markov chain Monte Carlo method.

$$\chi_{p,q}(A,B) = -\frac{i}{s(A,B)}\left[e^{ipA-iqB} - e^{-iqA+ipB} + e^{-ip(A+B)}\left(e^{-iqA} - e^{-iqB}\right) + e^{iq(A+B)}\left(e^{ipB} - e^{ipA}\right)\right] \tag{6.21}$$

where

$$s(A,B) = 8\sin\left(\frac{A-B}{2}\right)\sin\left(\frac{A+2B}{2}\right) \times \sin\left(\frac{2A+B}{2}\right) \tag{6.22}$$

The c_{pq} Fourier coefficients are then obtained using the following integral, proportional to the one given in [28]:

$$c_{pq} = e^{-6\beta}\int_{-\pi}^{\pi}\frac{d(A,B)s^2(A,B)}{24\pi^2}\overline{\chi}_{p,q}(A,B) \times \exp\left(2\beta\left(\cos A + \cos B + \cos\left(A+B\right)\right)\right) \tag{6.23}$$

Since this expression is more challenging to compute, we do not dwell on its analytical expression here.

7 Another Approach Involving Different Haar Integrals

We conclude this brief tour of pure lattice gauge theory with an alternate, perhaps more naive approach. Instead of using group characters, one might have instinctively expanded the exponential terms in the partition function, and used techniques such as Creutz's method to calculate integrals of products of group elements [21, 23], or Weingarten generating functions [24, 29]. This might be useful in higher dimensional non-abelian theories, where we have to integrate products of many matrix elements whose expressions are hard to come by or derive, although this avenue does not explicitly take advantage of the gauge group's structure. Let's define

$$U^{ijkl}_{\mu\nu}(n) = U_\mu(n)_{ij}U_\nu(n+\mu)_{jk}\overline{U}_\mu(n+\mu)_{lk}\overline{U}_\nu(n)_{il} \tag{7.1}$$

We then have:

$$\exp\left(\frac{2\beta}{g^2}\Re\left(\mathrm{Tr}\left[U_{\mu\nu}(n)\right]\right)\right) = \prod_{i,j,k,l}\sum_{p=0}^{\infty}\frac{\beta^p}{p!g^{2p}}\Re\left[U_{\mu\nu}^{ijkl}(n)\right]^p \tag{7.2}$$

It follows that

$$Z = C\int\mathcal{D}[U]\sum_{(p)\in\mathbb{N}^{kN^4}}\prod_{\substack{n\in\Lambda\\ \mu<\nu\\ i,j,k,l}}\frac{\beta^p}{p!g^{2p}}\Re\left[U_{\mu\nu}^{ijkl}(n)\right]^p \tag{7.3}$$

where $C = \exp\left(-d(d-1)n^d N\beta\right)$. Using the binomial theorem and expanding the real part yields

$$\begin{aligned} Z = C\sum_{(q)\le(p)}&\left(\prod_{\substack{n\in\Lambda\\ \mu<\nu\\ i,j,k,l}}\frac{\beta^p}{p!g^{2p}}\right)\\ \times\int\mathcal{D}[U]&\prod_{\substack{n\in\Lambda\\ \mu<\nu\\ i,j,k,l}}\binom{p}{q}U_{\mu\nu}^{ijkl}(n)^q\overline{U}_{\mu\nu}^{ijkl}(n)^{p-q}\end{aligned} \tag{7.4}$$

Let's define $f_{pq}^{ij}(U) = \binom{p}{q}^{1/4}U_{ij}^q\overline{U_{ij}}^{p-q}$, and add a bar to f when f takes a Hermitian conjugate in its argument.

$$\begin{aligned} Z = C\sum_{(q)\le(p)}&\left(\prod_{\substack{n\in\Lambda\\ \mu<\nu\\ i,j,k,l}}\frac{\beta^p}{p!g^{2p}}\right)\\ \times\int\mathcal{D}[U]&\prod_{\substack{n\in\Lambda\\ \mu<\nu\\ i,j,k,l}}f_{pq}^{ij}(U_\mu(n))f_{pq}^{jk}(U_\nu(n+\mu))\\ \times f_{pq}^{kl}&(U_\mu(n+\nu)^\dagger)f_{pq}^{li}(U_\nu(n)^\dagger)\end{aligned} \tag{7.5}$$

Reorganizing the products as before finally yields

$$\begin{aligned} Z = C\sum_{(q)\le(p)}&\left(\prod_{\substack{n\in\Lambda\\ \mu<\nu\\ i,j,k,l}}\frac{\beta^p}{p!g^{2p}}\right)\\ \times\prod_{\substack{n,\mu\\ i,j,k,l}}&\int\prod_{\nu=\mu+1}^{d-1}f_{pq(\mu,\nu,n)}^{ij}\overline{f}_{pq(\mu,\nu,n-\nu)}^{kl}\\ \times\prod_{\nu=0}^{\mu-1}&f_{pq(\nu,\mu,n)}^{jk}\overline{f}_{pq(\nu,\mu,n-\nu)}^{li}\end{aligned} \tag{7.6}$$

Admittedly, this is a convoluted expression. Nevertheless, it shows there exists an alternative to the formulae derived previously involving integrals of group elements, rather than group representation matrix elements.

8 A Word on MCMC Simulations

The numerical simulations used in this work largely depend on Markov Chain Monte Carlo methods (MCMC), in particular the Metropolis-Hastings algorithm [30, 31]. The general principle is outlined here in pseudocode (algorithm 1 below).

The simulations conducted for this work use 10^d lattice sites, $n_{iter} = 100000$ and $n_{thermal} = 90000$.

9 Conclusion

We started with an introduction to lattice gauge theory, and showed how it can benefit from techniques used to study the Ising model, such as representation-theoretic character expansions, dual lattices, and the mean field approximation. We also presented the construction of pure gauge lattice YM, before addressing specific gauge theories, both abelian and non-abelian. We've provided an extensive study of 1+1-dimensional lattice YM, and showed how one could construct tensor networks representing partition functions or Wilson loops. We've shown that

Data: $\beta \geq 0$, n_{iter}, n_{size}, n_{thermal}
Result: Expectation of observable O: $\langle O(\beta) \rangle$
initialize array *lattice* of shape $(n_{\text{size}}, \ldots, n_{\text{size}}, d, N, N)$ with random $SU(N)$ matrices chosen uniformly;
enforce boundary conditions;
initialize array a of size $(1, n_{\text{iter}} - n_{\text{thermal}})$ with zeros;
for $i \leftarrow 1$ **to** n_{iter} **do**
    choose random lattice site with uniform distribution $\mathbf{n}$ and direction j;
    update $lattice[\mathbf{n}, j]$ with a "slight perturbation";
    enforce boundary conditions;
    $\Delta S \leftarrow$ change in action ;
    accept update with probability $e^{-\Delta S}$;
    if $i > n_{thermal}$ **then**
        $a[i] \leftarrow O(lattice)$;
    end
end
the result is the mean of a

Algorithm 1: Metropolis algorithm calculating expectation value of an observable O

for $U(1)$ and $SU(N)$, the local error is subpolynomial in a power of β/D, where D is the bond dimension – this is perhaps the main result of this report. Incidentally, a similar result for $U(N)$ could be derived using the same reasoning as that used for $SU(N)$. Moreover, we've extensively studied an approximation where plaquettes are decoupled from one another, allowing us to derive analytical expressions for partition functions and Wilson loops, and obtain confinement.

Further work might focus on including fermions. Indeed, modern lattice QCD is heavily based on Monte Carlo simulations, which are limited by the sign problem when fermions are introduced, and tensor network methods could become very attractive in circumventing the former. Another interesting avenue for numerical approaches is the study of lattice gauge theory for quantum groups, which we've motivated in the simple case of $\mathbb{Z}_N$ for large N.

Furthermore, lattice gauge theory provides a promising way forward in non-perturbative and constructive quantum field theory. In particular, it can provide insightful results regarding phenomena such as confinement, asymptotic freedom and the mass gap problem. The coupling between lattice gauge theory and tensor networks is also mentioned in the literature as being useful to theories of quantum gravity. Indeed, furthering our understanding, both theoretical and numerical, of Yang-Mills theory on the lattice is a promising path forward in the study of general gauge theories. It might provide a much-needed deeper understanding of quantum field theory, and by extension pave the way for a quantum field-theoretic formulation of gravity.

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